Name: ______________________________

Teacher: ______________________________

Knox Grammar School

2015

Trial Higher School Certificate Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 100

Section I 10 Marks

- Answer Questions 1 to 10
- Use the Multiple Choice Answer Sheet

Section II 90 Marks

- Answer Questions 11 to 16
- All questions are worth 15 marks
- Answer each question in a separate Writing Booklet.

Subject Teachers
Mr M Vuletich
Mr I Bradford

Setter

Mr M Vuletich

This paper MUST NOT be removed from the examination room

Number of Students in Course: 35

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Section I

10 Marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 Which of the following is an expression for $\int \frac{1}{1+\sin x+\cos x}dx$?

(A) $\ln|t-1|+C$

(B) $\ln|t+1|+C$

(C) $\ln|t^2-1|+C$

(D) $\ln|t^2+1|+C$

2 What is the eccentricity for the hyperbola $\frac{y^2}{225}-\frac{x^2}{64}=1$?

(A) $\frac{8}{17}$

(B) $\frac{15}{17}$

(C) $\frac{17}{15}$

(D) $\frac{17}{8}$

3 The polynomial equation $x^3-5x^2+6=0$ has roots α, β and γ. Which of the following polynomial equations has roots α^2, β^2 and γ^2?

(A) $x^3-25x^2+60x-36=0$

(B) $x^3-25x^2+60x-12=0$

(C) $x^3-x^2+12x-36=0$

(D) $x^3-x^2+12x-12=0$

4 Given that $z-2+i$ is a factor of $P(z)=2z^3-7z^2+6z+5$ over the complex field, which one of the following statements must be true?

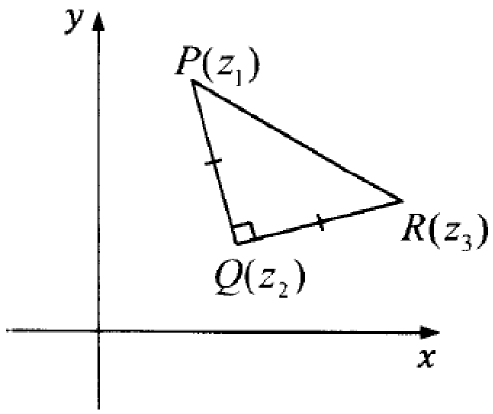
(A) $P(-2+i)=0$

(B) $P(-2-i)=0$

(C) The equation $P(z)=0$ has one complex and two real roots

(D) The equation $P(z)=0$ has one real and two complex roots

5 The vertices of ΔPQR are represented by the complex numbers z_1, z_2 and z_3 respectively. The ΔPQR is isosceles and right-angled at Q, as shown in the diagram.



Which of the following statements is true?

(A) $z_2-z_1=i(z_3-z_2)$

(B) $z_1-z_2=i(z_3-z_2)$

(C) $z_2-z_1=i(z_1-z_3)$

(D) $z_1-z_2=i(z_1-z_3)$

6 Using the binomial theorem $(1+x)^n = {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + ... + {}^nC_nx^n = \sum_{k=0}^{n} {}^nC_kx^k$, which of the following expressions is correct?

(A) $(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)...(n-k)}{n^k} \times \frac{1}{k!}$

(B) $(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)...(n-k)}{n^k} \times \frac{1}{(k+1)!}$

(C) $(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)...(n-k+1)}{n^k} \times \frac{1}{k!}$

(D) $(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)...(n-k+1)}{n^k} \times \frac{1}{(k+1)!}$

7 A particle of mass, m, travels with constant velocity, v, in a horizontal circle of radius, R, centre, C, around a track banked at an angle, α, to the horizontal, as shown in the diagram. There is no tendency for the particle to slip sideways.

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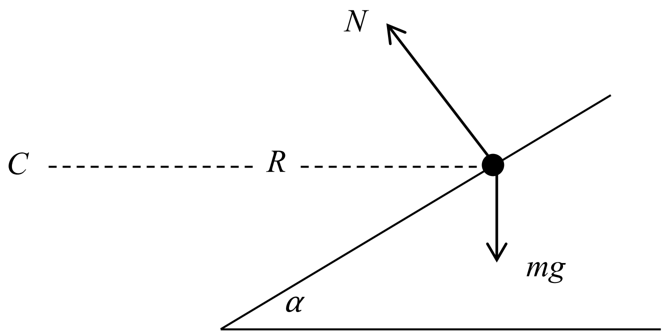


What is the expression for the vertical component of the forces acting on the particle?

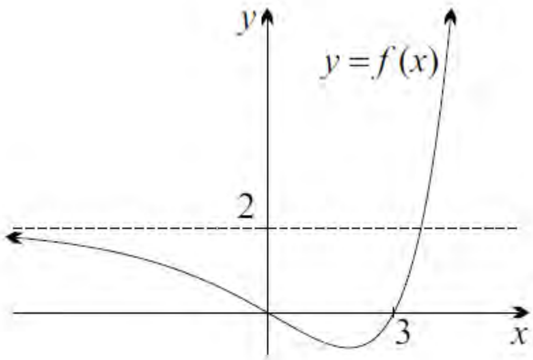
(A) $N\cos\alpha = mg$

(B) $N\sin\alpha = \frac{mv^2}{R}$

(C) $N\sin\alpha = mg$

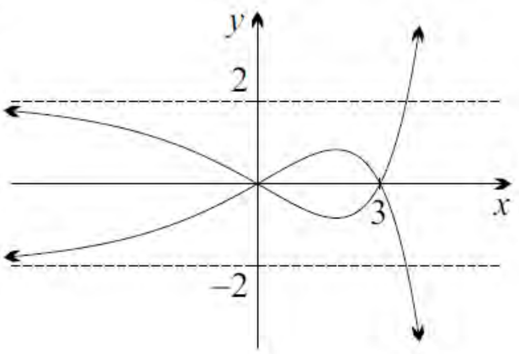
(D) $N\cos\alpha = \frac{mv^2}{R}$

8 The graph of $y = f(x)$ is shown below.

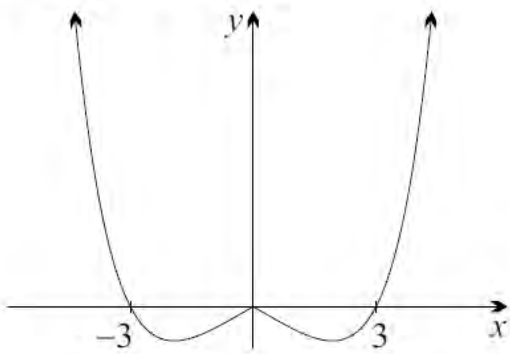


Which is the correct graph of $|y| = f(x)$?

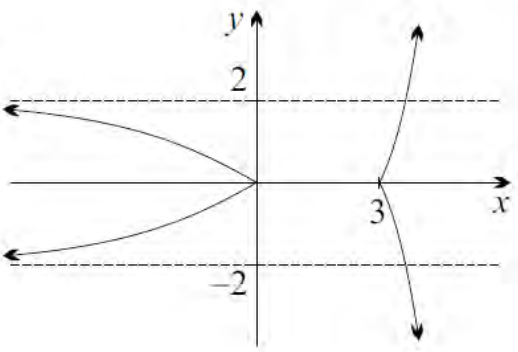
(A)



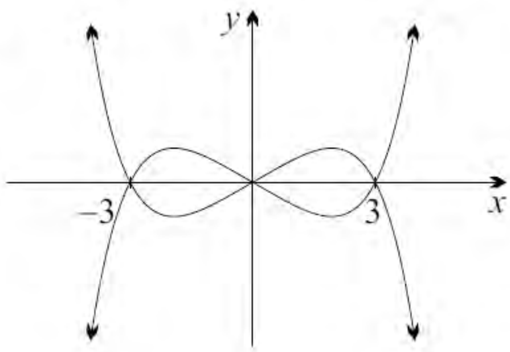
(B)



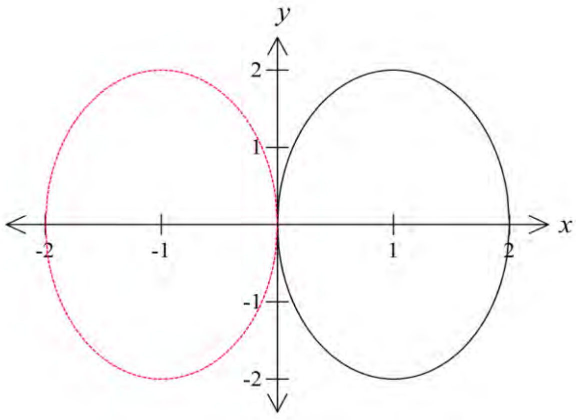
(C)



(D)



9 The region enclosed by the ellipse $(x-1)^2+\frac{y^2}{4}=1$ is rotated about the y axis to form a solid.



If slices are taken perpendicular to the axis of rotation, what is the correct expression for the volume?

(A) $V=\int_{-2}^{2}\pi\sqrt{1-y^2}\,dy$

(B) $V=\int_{-2}^{2}2\pi\sqrt{1-y^2}\,dy$

(C) $V=\int_{-2}^{2}\pi\sqrt{4-y^2}\,dy$

(D) $V=\int_{-2}^{2}2\pi\sqrt{4-y^2}\,dy$

10 How many six letter words can be formed using the letters of the word 'PRESSES'?

(A) 420

(B) 120

(C) 300

(D) 240

End of Section I

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let $z = 2 + 2i$ and $\omega = 1 + 3i$.

(i) Write $z - \omega$ in modulus-argument form. **2**

(ii) Find $\overline{\left(\dfrac{z}{\omega}\right)}$ in the form $a + bi$ where a and b real. **2**

(b) Evaluate $\int_1^e x^7 \ln x \, dx$. **3**

(c) Sketch the region in the complex plane where the inequalities

$|z - 1 - i| \le \sqrt{2}$ and $0 \le \arg(z - 1 - i) \le \dfrac{\pi}{4}$ are satisfied simultaneously. **3**

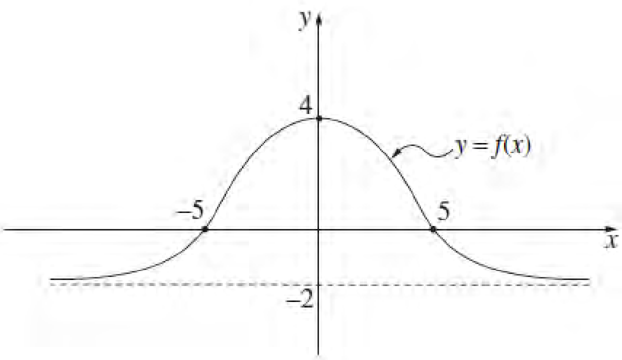
(d) Without the use of calculus, sketch the graph of $y = \dfrac{x^3 + 1}{x}$, showing any asymptotes and intercepts with the coordinate axes. **2**

(e) The area defined by $0 \le y \le \sin x$ for $0 \le x \le \pi$ is rotated about the y-axis. Use the method of cylindrical shells to find the volume of the resulting solid of revolution. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) The diagram shows the graph of $y = f(x)$.

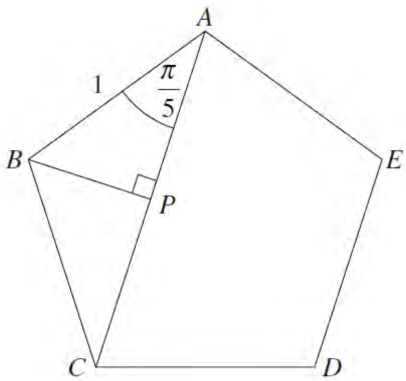


Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = \dfrac{1}{\sqrt{f(x)}}$ **2**

(ii) $y = x f(x)$ **2**

(b)



In the diagram, *ABCDE* is a regular pentagon with sides of length 1 unit. The perpendicular to *AC* through *B* meets *AC* at *P*.

Copy or trace the diagram into your writing booklet.

(i) Let $u = \cos\dfrac{\pi}{5}$. Use the cosine rule in ΔACD to show that $8u^3 - 8u^2 + 1 = 0$. **2**

(ii) One root of $8x^3 - 8x^2 + 1 = 0$ is $\dfrac{1}{2}$. By finding the other roots of $8x^3 - 8x^2 + 1 = 0$, find the exact value of $\cos\dfrac{\pi}{5}$. **2**

Question 12 continues on page 9

Question 12 (continued)

(c) Find the equation of the tangent to the curve $x^2 - xy + y^3 = 5$ at the point $(2,-1)$. **3**

(d) Let $I_n = \int_0^1 \frac{x^{2n}}{x^2+1}\,dx$ for integers $n \geq 0$.

(i) Show that $I_n + I_{n-1} = \frac{1}{2n-1}$ for integers $n \geq 1$. **2**

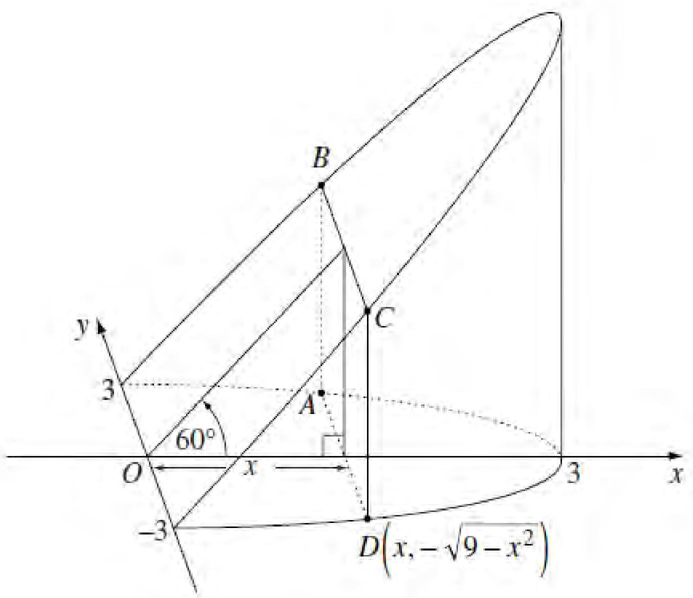
(ii) Hence, or otherwise, find $\int_0^1 \frac{x^4}{x^2+1}\,dx$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Find $\int \frac{5x^2 - 3x + 13}{(x-1)(x^2+4)} dx$. **3**

(b) The base of a right cylinder is a circle in the *xy* plane with centre *O* and radius 3. A wedge is obtained by cutting this cylinder with the sloping plane through the *y*-axis inclined at 60° to the *xy* plane, as shown in the diagram.



A rectangular slice *ABCD* is taken perpendicular to the base of the wedge at a distance *x* from the *y*-axis.

(i) Show that the area of *ABCD* is given by $2x\sqrt{27-3x^2}$. **2**

(ii) Find the volume of the wedge. **2**

Question 13 continues on page 11

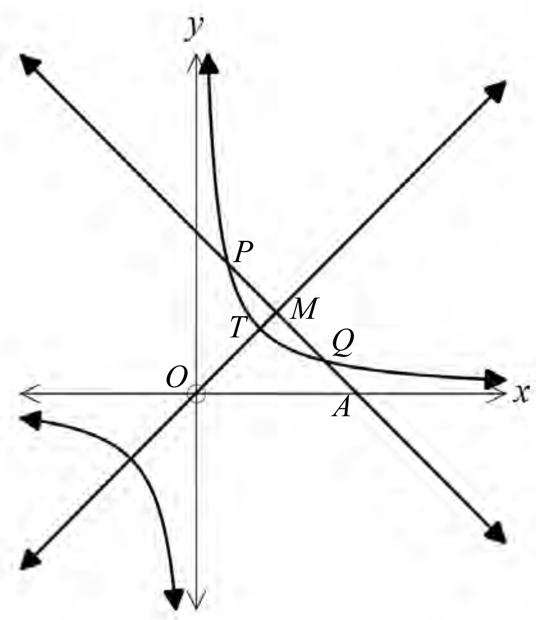
Question 13 (continued)

(c) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $0<b<a$, cuts the x-axis at A and A'.
The ellipse has eccentricity e and $S\left(ae,0\right)$ is the focus of ellipse nearer to A.
The focal chord PSQ is perpendicular to the x-axis.

(i) Draw a diagram to represent this information. **1**

(ii) Show that $\frac{1}{AS}+\frac{1}{A'S}=\frac{4}{PQ}$. **2**

(d) In the diagram $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ where $0<p<q$,
are points on the hyperbola $xy=c^2$.
M is the midpoint of PQ and the line PQ cuts the x-axis at A.
OM cuts the hyperbola at T.



(i) Show that gradient $OM = -1\times$ gradient MA . **2**

(ii) Hence show that $OM = MA$. **1**

(iii) Show that the tangent to the hyperbola at T is parallel to the chord PQ. **2**

End of Question 13

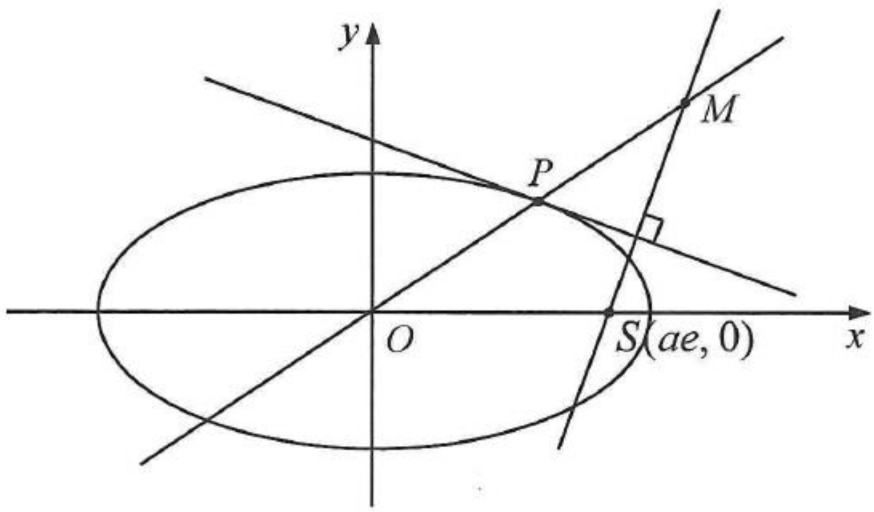
Question 14 **(15 marks)** Use a SEPARATE writing booklet

(a) The polynomial $P(x) = px^3 - 3qx + r$ has a zero multiplicity 2. **3**

Show that $4q^3 = pr^2$.

(b) The diagram shows the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with focus $S(ae, 0)$ and centre O.

$P(a\cos\theta, b\sin\theta)$ is any point on the ellipse. The line through S perpendicular to the tangent at P and the line OP produced meet at M.



(i) Show the gradient of the tangent at P is given by $-\frac{b\cos\theta}{a\sin\theta}$. **1**

(ii) Show that M lies on the corresponding directrix to the focus at S. **3**

(c) A group of 30 students is to be divided into three groups consisting of 7, 8 and 15 students. In how many ways can this be done? Leave your answer in unsimplified form. **1**

Question 14 continues on page 13

Question 14 (continued)

(d) An object of mass 2 kg is projected vertically upwards from ground level at a speed of 20 m/s. It experiences a resistance of $\frac{v^2}{2}$ Newtons at a speed of v m/s, and reaches a maximum height of H metres. Take upwards as positive and g = 10m/s^2.

(i) If x is the displacement of the object from ground level after t seconds, show that its acceleration is given by $\ddot{x} = \frac{-40 - v^2}{4}$. **1**

(ii) Show that it takes approximately 0.8 seconds for the object to reach its maximum height. **2**

(iii) Find the maximum height H reached by the object. **2**

(iv) Calculate the speed of projection required to reach a maximum height of $2H$ metres. **2**

End of Question 14

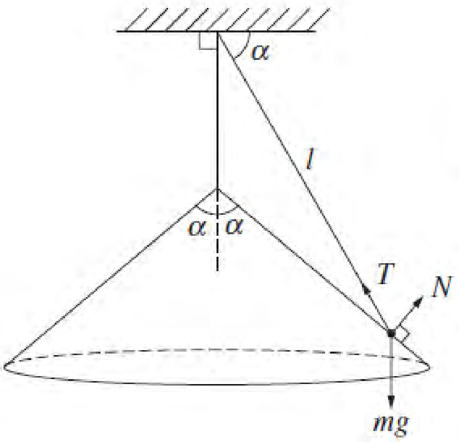
Question 15 **(15 marks)** Use a SEPARATE writing booklet

(a) (i) Use calculus to show that $x > \ln(1+x)$ for all $x > 0$. **2**

(ii) Use the inequality in part (i) and the principle of mathematical induction to prove that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} > \ln(1+n)$$ for all positive integers n. **3**

(b) A particle of mass m is suspended by a string of length l from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity ω. The angle of the cone at its vertex is 2α, where $\alpha > \frac{\pi}{4}$, and the string makes an angle of α with the horizontal as shown in the diagram. The forces acting on the particle are the tension in the string T, the normal reaction to the cone N and the gravitational force mg.



NOT TO SCALE

(i) Show, with the aid of a diagram, that the vertical component of N is $N\sin\alpha$. **1**

(ii) Show that $T + N = \frac{mg}{\sin\alpha}$, and find an expression for $T - N$ in terms of m, l and ω. **3**

(iii) The angular velocity is increased until $N = 0$, that is, when the particle is about to lose contact with the cone. Find an expression for this value of ω in terms of α, l and g. **2**

Question 15 continues on page 15

Question 15 (continued)

(c) (i) It can be shown that for positive real numbers a and b that

$$a^2 + b^2 \geq 2ab \quad \text{(DO NOT PROVE THIS).}$$

Hence show for positive real numbers a, b, c and d that:

$$3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$$ **2**

(ii) Hence show for positive real numbers a, b, c and d that

if $a + b + c + d = 1$ then $ab + ac + ad + bc + bd + cd \leq \frac{3}{8}$. **2**

End of Question 15

Question 16 **(15 marks)** Use a SEPARATE writing booklet

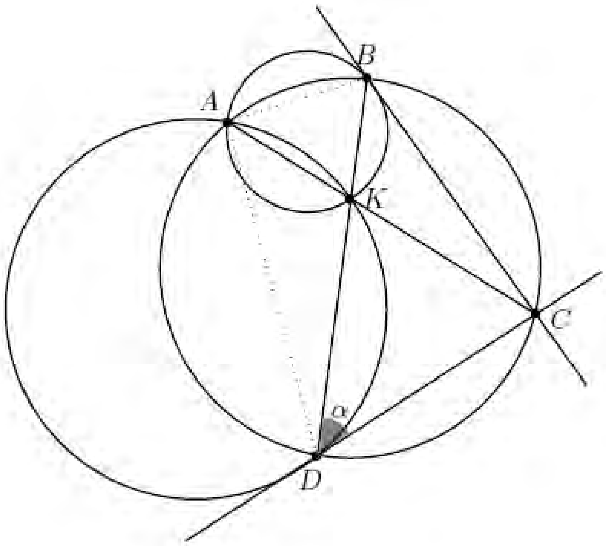
(a) There are 40 balls in a game of lotto numbered from 1 to 40.
Six balls are selected at random.

(i) What is the probability, correct to four decimal places, of drawing the number 7 in exactly two of the next five games? **2**

(ii) What is the probability, correct to four decimal places, of drawing the number 7 in at least two of the five games? **1**

(iii) What is the probability that the number 7 is drawn and it is the highest number drawn in at least one of the next five games? Give your answer in scientific notation correct to three significant figures. **2**

(b)



In the diagram above, *ABCD* is a cyclic quadrilateral and diagonals *AC* and *BD* intersect at *K*. Circles *AKD* and *AKB* are drawn and it is known that *CD* is a tangent to circle *AKD*. Let $\angle CDB = \alpha$.

Use the separate blue answer sheet for Question 16 (b).

(i) Prove that ΔBCD is isosceles. **2**

(ii) Prove that *CB* is a tangent to circle *AKB* **2**

Question 16 is continued on page 17

Question 16 (continued)

(c) Suppose that x is a positive real number.

(i) Show that $\frac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + ... + t^{4n} < \frac{1}{1+t^2} + t^{4n+2}$, for $0 < t < x$. **3**

(ii) Hence show that:

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + \frac{x^{4n+1}}{4n+1} < \tan^{-1} x + \frac{x^{4n+3}}{4n+3}$$ **1**

(iii) Explain why $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$ for $0 \le x \le 1$. **1**

(iv) Deduce that $\pi = 4 \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...)$. **1**

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2015 MATHEMATICS EXTENSION 2 TRIAL HSC SOLUTIONS

1.	B	2.	C	3.	A	4.	D	5	B
6.	C	7.	A	8.	C	9.	D	10.	A

Question 1

$$\int \frac{1}{1+\sin x+\cos x}\,dx = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$$

$$= \int \frac{2}{1+t^2+2t+1-t^2}\,dt$$

$$= \int \frac{1}{1+t}\,dt$$

$$= \ln|t+1| + c \qquad \therefore B$$

Question 2

For $\frac{y^2}{225} - \frac{x^2}{64} = 1$

$a = 8,\ b = 15$

$a^2 = b^2(e^2-1)$

$\therefore e^2 = 1 + \frac{64}{225}$

$e = \frac{17}{15}$ $\qquad \therefore C$

Question 3

Let $x = \sqrt{y}$ for roots $\alpha^2, \beta^2, \gamma^2$

$\therefore (\sqrt{y})^3 - 5(\sqrt{y})^2 + 6 = 0$

$y^{\frac{3}{2}} = 5y - 6$

$\therefore y^3 = 25y^2 - 60y + 36$

$\therefore y^3 - 25y^2 + 60y - 36 = 0$ $\qquad \therefore A$

Question 4

$P(z)$ has real co-efficients $\therefore$ complex roots occur in conjugate pairs. Since $P(z)$ is of degree 3, it must have 2 complex and one real root $\therefore D$

Question 5

$\vec{QP} = i \times \vec{QR}$ as $|QP| = |QR|$ and $\angle PQR = 90°$ and QR is rotated anti-clockwise about Q

$\therefore z_1 - z_2 = i(z_3 - z_2)$ $\qquad \therefore B$

Question 6

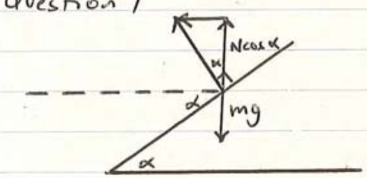
$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} {}^nC_k \left(\frac{1}{n}\right)^k$$

$$= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\cdot\frac{1}{n^k}$$

$$= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\ldots 3\times 2\times 1}{n^k\,(n-k)(n-k-1)\times 3\times 2\times 1}\cdot\frac{1}{k!}$$

$$= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k}\cdot\frac{1}{k!} \qquad \therefore C$$

Question 7



Resolving vertical forces

$N\cos\alpha = mg$ $\qquad \therefore A$

Question 8

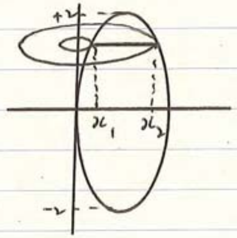
If $|y| = f(x)$ then $f(x) \geqslant 0$

Apply definition $|y| = \sqrt{y^2}$

$\sqrt{y^2} = f(x)$

$y = \pm f(x)$ only for $f(x) \geqslant 0$ $\therefore C$

Question 9



$\Delta V = \pi (R+r)(R-r) \cdot \delta y$

where $R = x_2$, $r = x_1$

Taking x_2, x_1 as roots of equation $(x-1)^2 + \frac{y^2}{4} = 1$ at some fixed height y above x-axis:

$4(x^2 - 2x + 1) + y^2 - 4 = 0$

$4x^2 - 8x + y^2 = 0$ has roots x_1, x_2

$x_1 + x_2 = 2$

$x_2 - x_1 = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$ $\quad x_2 - x_1 > 0$

$= \sqrt{4 - 4 \cdot \frac{y^2}{4}}$

$= \sqrt{4 - y^2}$

$\therefore V = 2\pi \int_{-2}^{2} \sqrt{4 - y^2}\, dy$ $\quad \therefore D$

Question 10

Using cases:

Omit S : $\frac{6!}{2! \times 2!} = 180$ (2 Es, 2 Ss)

Omit E : $\frac{6!}{3!} = 120$ (3 Ss)

$\therefore$ Total = 420

Omit P : $\frac{6!}{3! \times 2!} = 60$ (2 Es, 3 Ss)

$\therefore A$

Omit R $\frac{6!}{3! \times 2!} = 60$ (2 Es, 3 Ss)

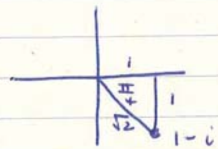
Section II

Question 11

(a) (i) $z - w = 2 + 2i - (1 + 3i)$

$= 1 - i$ ✓

$= \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$ ✓



(ii) $\overline{\left(\frac{z}{w}\right)} = \overline{\left(\frac{2+2i}{1+3i} \times \frac{1-3i}{1-3i}\right)}$

$= \overline{\left(\frac{2 - 4i + 6}{10}\right)}$

$= \overline{\frac{4}{5} - \frac{2}{5}i}$ ✓

$= \frac{4}{5} + \frac{2}{5}i$ ✓

(b) $I = \int_1^e x^7 . \ln x \, dx$ $\quad u = \ln x \quad v' = x^7$, $u' = \frac{1}{x} \quad v = \frac{x^8}{8}$ ✓

$= \left[\frac{x^8}{8} \ln x\right]_1^e - \frac{1}{8}\int_1^e x^7\, dx$ ✓

$= \frac{e^8}{8} - \frac{1}{8}\left[\frac{x^8}{8}\right]_1^e$

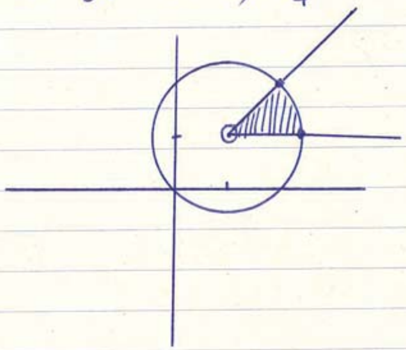
$= \frac{e^8}{8} - \frac{1}{8}\left(\frac{e^8}{8} - \frac{1}{8}\right)$

$= \frac{7e^8 + 1}{64}$ ✓ or equivalent

(c) $|z-(1+i)| \leq 2$

$0 \leq \arg(z-(1+i)) \leq \frac{\pi}{4}$

✓ circle
✓ sector
✓ shaded region



(d) $y = \frac{x^3+1}{x}$

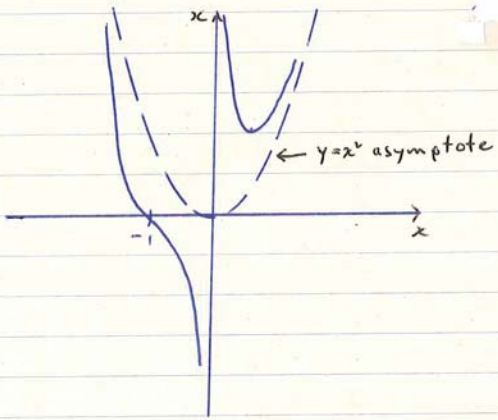
$= x^2 + \frac{1}{x}$

At $x=0$, y undefined
$y=0$, $x=-1$

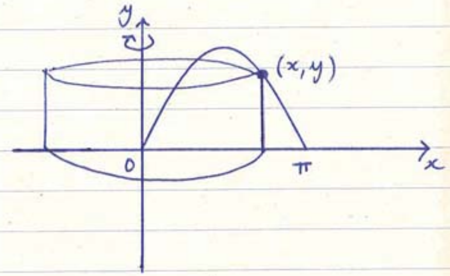
$\therefore$ Vertical Assymptote at $x=0$

as $x \to 0^+$, $y \to \infty$
$x \to 0^-$, $y \to -\infty$

as $x \to \infty$ $y \to (x^2)^+$
$x \to -\infty$ $y \to (x^2)^-$



(e)



$\therefore \delta V = 2\pi x y \delta x$

$\therefore V = 2\pi \int_0^\pi xy\, dx$

$= 2\pi \int_0^\pi x \sin x\, dx$

$u = x$ $v' = \sin x$
$u' = 1$ $v = -\cos x$

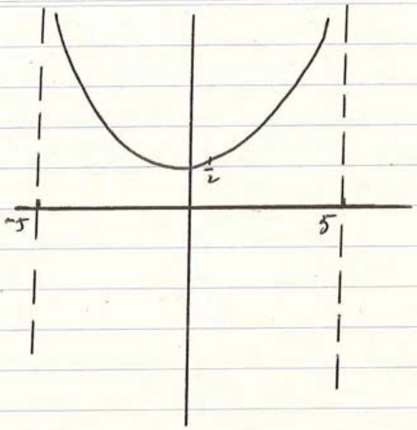
$= 2\pi \left\{ \left[-x\cos x\right]_0^\pi + \int_0^\pi \cos x\, dx \right.$

$= 2\pi \left\{ \pi + \left[\sin x\right]_0^\pi \right\}$

$= 2\pi^2$ units^3

Question 12

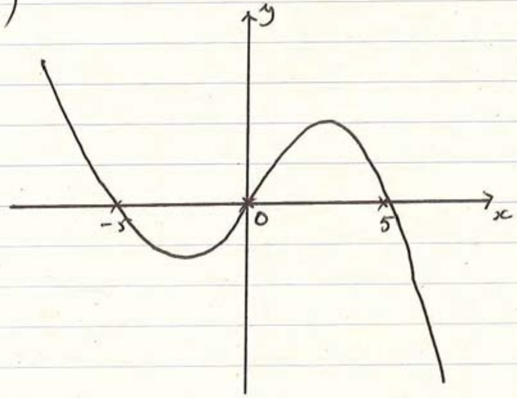
(a) i)



✓ asymptotes and intercept

✓ shape.

ii)



✓ intercepts

✓ shape.

(b) (i) $\cos\frac{\pi}{5} = AP$

$\therefore AP = u$

$\Delta APB \equiv \Delta CPB$ (RHS)

$\therefore PC = u$

$\therefore AC = 2u$

$\angle DAE = \frac{\pi}{5}$ (corresponding angle of congruent Δ's ABC and AED (SAS))

$\therefore \angle CAD = \frac{3\pi}{5} - \frac{\pi}{5} - \frac{\pi}{5}$ (Interior $\angle$ of regular pentagon is $\frac{3\pi}{5}$)

$= \frac{\pi}{5}$

progress to setup.

$\therefore CD^2 = AC^2 + AD^2 - 2\times AC \times AD \times \cos\frac{\pi}{5}$

$1 = (2u)^2 + (2u)^2 - 2(2u)(2u)\cos\frac{\pi}{5}$

$1 = 4u^2 + 4u^2 - 8u^3$ as $\cos\frac{\pi}{5} = u$

$\therefore 8u^3 - 8u^2 + 1 = 0$ as required

✓ correct use of cosine rule.

ii) So $\frac{1}{2} + \alpha + \beta = 1$ (sum of roots)

$\alpha + \beta = \frac{1}{2}$

$\frac{1}{2}\alpha\beta = -\frac{1}{8}$ (product of roots)

$\alpha\beta = -\frac{1}{4}$

$\therefore \alpha - \frac{1}{4\alpha} = \frac{1}{2}$

✓ significant progress

$4\alpha^2 - 1 = 2\alpha$

$4\alpha^2 - 2\alpha - 1 = 0$

$\alpha = \frac{2 \pm \sqrt{4+16}}{8}$

$= \frac{1 \pm \sqrt{5}}{4}$

✓

$\therefore \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ as $\cos\frac{\pi}{5} > 0$

(c) $\frac{d}{dx}x^2 - \frac{d}{dx}xy + \frac{d}{dx}y^3 = 5$

$2x - \left(x \cdot \frac{dy}{dx} + y\right) + 3y^2 \frac{dy}{dx} = 0$

$2x - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$ ✓

$\frac{dy}{dx}(3y^2 - x) = y - 2x$

$\therefore \frac{dy}{dx} = \frac{y-2x}{3y^2-x}$ ✓

at $(2,-1)$ $\frac{dy}{dx} = \frac{-1-4}{3-2} = -5$

$\therefore y+1 = -5(x-2)$

$y+1 = -5x+10$

$5x+y-9=0$ is tangent ✓

(d) i) $I_n + I_{n-1} = \int_0^1 \frac{x^{2n}}{x^2+1}dx + \int_0^1 \frac{x^{2(n-1)}}{x^2+1}dx$

$= \int_0^1 \frac{x^{2n} + x^{2n-2}}{x^2+1}dx$

$= \int_0^1 \frac{x^{2n} + \frac{x^{2n}}{x^2}}{x^2+1}dx$

$= \int_0^1 \frac{x^2x^{2n} + x^{2n}}{x^2(x^2+1)}dx$

$= \int_0^1 \frac{x^{2n}}{x^2}dx$

$= \int_0^1 x^{2n-2}dx$

✓ significant progress

$= \left[\frac{x^{2n-1}}{2n-1}\right]_0^1$ ✓

$= \frac{1}{2n-1}$

(d) ii) $\int_0^1 \frac{x^4}{x^2+1}dx = I_2$

$= \frac{1}{2(2)-1} - I_1$

$= \frac{1}{3} - \left[\frac{1}{2(1)-1} - I_0\right]$

$= \frac{1}{3} - 1 + I_0$ and $I_0 = \int_0^1 \frac{1}{x^2+1}dx$

$= \frac{\pi}{4} - \frac{2}{3}$ ✓ $= \tan^{-1}x\Big]_0^1$

$= \frac{\pi}{4}$ ✓

Question 13

(a) let $\frac{5x^2-3x+13}{(x-1)(x^2+4)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

$\equiv \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$

$\therefore 5x^2-3x+13 = A(x^2+4)+(Bx+C)(x-1)$ ✓

$x=1$ $\therefore 15 = 5A$

$A = 3$

$x=0$ $\therefore 13 = 12 - C$

$\therefore C = -1$

$x=-1$ $\therefore 21 = 15 + (-B-1)\cdot -2$

$6 = 2B+2$

$\therefore B = 2$ ✓

$\therefore I = \int \frac{3}{x-1} + \frac{2x-1}{x^2+4}dx$

$= 3\ln|x-1| + \int\frac{2x}{x^2+4}dx - \int\frac{1}{x^2+4}dx$

$= 3\ln|x-1| + \ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ ✓

or equivalent

(b) (i) Area = AD × CD

$= 2y \times CD$ $\tan 60^\circ = \frac{CD}{x}$

$= 2y \times \sqrt{3}x$ ✓ $\therefore CD = \sqrt{3}x$ } ✓

$= 2\sqrt{9-x^2} \cdot \sqrt{3}x$

$= 2x\sqrt{3(9-x^2)}$

$= 2x\sqrt{27-3x^2}$

ii) $\delta V = 2x\sqrt{27-3x^2} \cdot \delta x$

$\therefore V = \int_0^3 2x(27-3x^2)^{\frac{1}{2}}\,dx$

$= -\frac{1}{3}\int_0^3 -6x(27-3x^2)^{\frac{1}{2}}\,dx$ ✓

$= -\frac{1}{3}\left[\frac{(27-3x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$

$= -\frac{2}{9}\left[0 - 27^{\frac{3}{2}}\right]$

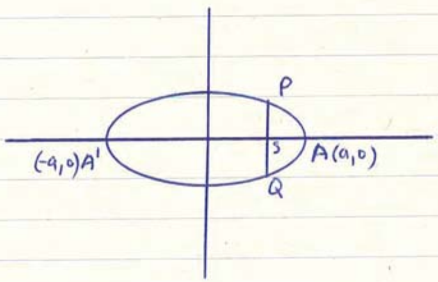
$= \frac{2}{9} \times (\sqrt{27})^3$

$= \frac{2}{9} \times (3\sqrt{3})^3$

$= \frac{2}{9} \times 27 \times 3\sqrt{3}$

$= 18\sqrt{3}$ units^3 ✓

(c) i)



✓ diagram

ii) $L.H.S = \frac{1}{AS} + \frac{1}{A'S}$

$= \frac{1}{a-ae} + \frac{1}{ae+a}$

$= \frac{ae+a+a-ae}{a^2-a^2e^2}$

$= \frac{2a}{a^2(1-e^2)} = \frac{2a}{b^2}$ ✓   For ellipse $\frac{b^2}{a^2} = 1-e^2$

$R.H.S = \frac{4}{PQ}$ at $x = ae$ $\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$

$= \frac{4}{2b\sqrt{1-e^2}}$ $\therefore y^2 = b^2(1-e^2)$

$= \frac{4}{2b} \cdot \frac{1}{\frac{b}{a}}$ $y = \pm b\sqrt{1-e^2}$

$= \frac{4a}{2b^2}$ ✓ $\therefore Q(ae, -b\sqrt{1-e^2})$

$= \frac{2a}{b^2}$ $P(ae, b\sqrt{1-e^2})$

$\therefore \frac{1}{AS} + \frac{1}{A'S} = \frac{4}{PQ}$

(d) i) $M\left(\frac{cp+cq}{2}, \frac{\frac{c}{p}+\frac{c}{q}}{2}\right)$

$= M\left(\frac{c}{2}(p+q), \frac{c}{2}\left(\frac{p+q}{pq}\right)\right)$

$\therefore m_{OM} = \frac{\frac{c}{2}\left(\frac{p+q}{pq}\right)}{\frac{c}{2}(p+q)} = \frac{1}{pq}$ ✓

$m_{MA} = m_{PQ} = \frac{\frac{c}{q}-\frac{c}{p}}{cq-cp} = \frac{c(p-q)}{pq} \Big/ c(q-p)$

$= -\frac{1}{pq}$ ✓

$\therefore m_{OM} = -1 \times m_{MA}$

ii) $\tan \angle MOA = m_{OM} = \frac{1}{pq}$

Since $\tan \angle MAx = m_{AM} = -\frac{1}{pq}$

$\therefore \angle MAx = \pi - \angle MOA$

$\therefore \angle MAO = \angle MOA$

$\therefore OM = MA$ (Equal sides of Isosceles ΔOMA)

iii) Equation of OM is $y = \frac{1}{pq}x$ — ①

To find coordinates of T:

$xy = c^2$ — ②

$x \cdot \frac{1}{pq} x = c^2$

$x^2 = c^2 pq$

$x = c\sqrt{pq}$

$y = c^2 x^{-1}$

$y' = \frac{-c^2}{x^2}$

at $x = c\sqrt{pq}$ $\quad m_T = \frac{-c^2}{(c\sqrt{pq})^2} = -\frac{1}{pq} = m_{MA} = m_{PQ}$

$\therefore$ Tangent at T // chord PQ.

Question 14

(a) $P(x) = px^3 - 3qx + r$

$P'(x) = 3px^2 - 3q$

Double root is also a root of $P'(x) = 0$

$\therefore 3px^2 - 3q = 0$

$x^2 = \frac{q}{p}$ ✓

$x = \frac{\pm\sqrt{q}}{\sqrt{p}}$ are possible double roots.

$P\left(\frac{\sqrt{q}}{\sqrt{p}}\right) = p \cdot \left(\frac{\sqrt{q}}{\sqrt{p}}\right)^3 - 3q \cdot \frac{\sqrt{q}}{\sqrt{p}} + r = 0$

$\therefore p \cdot \frac{q\sqrt{q}}{p\sqrt{p}} - 3q \cdot \frac{\sqrt{q}}{\sqrt{p}} + r = 0$

$\times \sqrt{p}$ $\quad q\sqrt{q} - 3q\sqrt{q} + r\sqrt{p} = 0$

$\therefore r\sqrt{p} = 2q\sqrt{q}$

squaring $\quad r^2 p = 4q^3$

Progress

(b) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$ ✓

at $P(a\cos\theta, b\sin\theta)$ $m_T = \frac{-b^2 . a\cos\theta}{a^2 . b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$

(ii) $m_{SM} = \frac{a\sin\theta}{b\cos\theta}$ (MS ⊥ tangent at P)

$\therefore y - y_0 = m(x - x_0)$

$y - 0 = \frac{a\sin\theta}{b\cos\theta}(x - ae)$

$\therefore y = \frac{a\sin\theta}{b\cos\theta}(x - ae)$ is equation of MS

Equation of OP is $y = mx$ with $m_{OP} = \frac{b\sin\theta}{a\cos\theta}$

ie, $y = \frac{b\sin\theta}{a\cos\theta}x$

Solving simultaneously,

$\frac{b\sin\theta}{a\cos\theta}x = \frac{a\sin\theta}{b\cos\theta}(x - ae)$

$\frac{b^2}{a^2}x = x - ae$

$\therefore x\left(\frac{b^2}{a^2} - 1\right) = -ae$ as $\frac{b^2}{a^2} = 1 - e^2$

$x = \frac{-ae}{1 - e^2 - 1} = \frac{-ae}{-e^2} = \frac{a}{e}$ $\therefore$ M lies on $S(\frac{a}{e}, 0)$

(c) ${}^{30}C_7 \times {}^{23}C_8 \times {}^{15}C_{15} = {}^{30}C_7 \times {}^{23}C_8$

or ${}^{30}C_{15} \times {}^{15}C_8 \times {}^{7}C_7 = {}^{30}C_{15} \times {}^{15}C_8$

or Equivalent

(d) i) $F = -mg - \frac{v^2}{2}$

$\therefore m\ddot{x} = -mg - \frac{v^2}{2}$ and $m = 2, g = 10$

$\therefore 2\ddot{x} = -20 - \frac{v^2}{2}$ ✓

$\ddot{x} = -10 - \frac{v^2}{4}$

$= \frac{-40 - v^2}{4}$

ii) $\frac{dv}{dt} = \frac{-40 - v^2}{4}$

$\therefore \frac{dt}{dv} = \frac{-4}{40 + v^2}$

$t = -4 \cdot \frac{1}{\sqrt{40}} \tan^{-1}\left(\frac{v}{\sqrt{40}}\right) + c$ ✓

at $t = 0$, $v = 20$ $\therefore 0 = \frac{-2}{\sqrt{10}} \tan^{-1}\left(\frac{20}{2\sqrt{10}}\right) + c$

$\therefore c = \frac{2}{\sqrt{10}} \tan^{-1}\left(\frac{10}{\sqrt{10}}\right)$

$\therefore t = \frac{2}{\sqrt{10}}\left(\tan^{-1}\left(\frac{10}{\sqrt{10}}\right) - \tan^{-1}\left(\frac{v}{\sqrt{40}}\right)\right)$

at max height $v = 0$ $\therefore t = \frac{2}{\sqrt{10}} \tan^{-1}\left(\frac{10}{\sqrt{10}}\right)$ ✓

$= 0.799\ldots$

$\doteqdot 0.8$ seconds

iii) $v \cdot \frac{dv}{dx} = -\frac{40 - v^2}{4}$

$\therefore \frac{dx}{dv} = \frac{-4v}{40 + v^2}$

$x = -2\int \frac{2v}{40+v^2}\,dv$

$= -2\ln(40+v^2) + C$

at $x = 0$, $v = 20m$ $\therefore C = 2\ln 440$

$\therefore x = 2\ln\left(\frac{440}{40+v^2}\right)$ ✓

at $v = 0$, $x = H$ (max height

$\therefore H = 2\ln 11$ ✓

iv) let speed of projection be u

$\therefore x = 2\ln\left(\frac{40+u^2}{40+v^2}\right)$ from part iii) ✓

at $v = 0$, $x = 2H = 4\ln 11$

$\therefore 4\ln 11 = 2\ln\left(\frac{40+u^2}{40}\right)$

$\therefore 121 = \frac{40+u^2}{40}$

$u^2 = 4800$

$u = 40\sqrt{3}$ m/s $u > 0$ ✓

$\doteqdot 69.282\ldots$ m/s

Question 15

a) i) let $f(x) = x - \ln(1+x)$

$f'(x) = 1 - \frac{1}{1+x} > 0$ for all $x > 0$ ✓

as $1 + x > 1$

and $f(0) = 0$

$\therefore f(x) > 0$ as $f(x)$ is monotonic increasing for all $x > 0$ and $f(0) = 0$ ✓

$\therefore x - \ln(1+x) > 0$

$\therefore x > \ln(1+x)$

ii) When $n = 1$

L.H.S $= \frac{1}{1}$ $= 1$ R.H.S $= \ln(1+1)$ $= \ln 2$ $\doteqdot 0.69\ldots$ ✓

$\therefore$ statement is true when $n = 1$

Assume statement true for $n = k$

ie, $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} > \ln(1+k)$

When $n = k+1$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} + \frac{1}{k+1} - \ln(1+k+1)$

$> \ln(1+k) - \ln(2+k) + \frac{1}{k+1}$ by assumption ✓

$> \ln(1+k) - \ln(2+k) + \ln\left(1 + \frac{1}{k+1}\right)$ by part i)

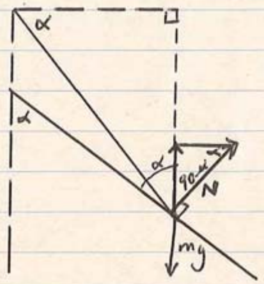
$= \ln\left(\frac{1+k}{2+k}\right) + \ln\left(\frac{k+2}{k+1}\right)$ ↓ progress ✓

$= \ln\left(\frac{1+k}{2+k} \cdot \frac{k+2}{k+1}\right)$

$= \ln 1 = 0$

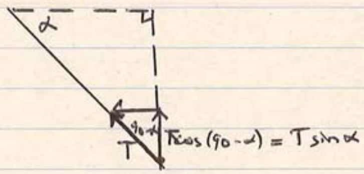
Question 15 Continued

i)



From diagram: Vertical component of N is $N\cos(90-\alpha) = N\sin\alpha$.

ii)



Vertical component of T is $T\sin\alpha$ ✓

Resolving forces vertically:

$$T\sin\alpha + N\sin\alpha = mg. \quad ✓$$

$$\therefore T + N = \frac{mg}{\sin\alpha} \quad — ①$$

Sum of radial forces $= mr\omega^2$

$$\therefore T\cos\alpha - N\cos\alpha = mr\omega^2 \quad \text{but } \cos\alpha = \frac{r}{l}$$

$$\therefore T - N = ml\omega^2 — ② \quad ✓$$

iii) ① − ②: $$2N = \frac{mg}{\sin\alpha} - ml\omega^2 \quad ✓$$

at $N = 0$ $$0 = \frac{g}{\sin\alpha} - l\omega^2 \quad ✓$$

$$\therefore \omega = \sqrt{\frac{g}{l\sin\alpha}} \quad \omega > 0$$

Question 15

(c) i)

$$\left.\begin{array}{l} a^2 + b^2 \geq 2ab \\ a^2 + c^2 \geq 2ac \\ a^2 + d^2 \geq 2ad \\ b^2 + c^2 \geq 2bc \\ b^2 + d^2 \geq 2bd \\ c^2 + d^2 \geq 2cd \end{array}\right\} ✓$$

Adding $$3a^2 + 3b^2 + 3c^2 + 3d^2 \geq 2ab + 2ac + 2ad + 2bc + 2bd + 2cd \quad ✓$$

$$3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd).$$

ii) Now $$\Sigma a^2 = (\Sigma a)^2 - 2(\Sigma ab)$$

So

$$3\left[(a+b+c+d)^2 - 2(ab+ac+ad+bc+bd+cd)\right] \geq 2(ab+ac+ad+bc+bd+cd) \quad ✓$$

$$\therefore 3(1)^2 \geq 8(ab+ac+ad+bc+bd+cd) \quad ✓ \quad \text{as } a+b+c+d=1$$

$$\therefore ab + ac + ad + bc + bd + cd \leq \frac{3}{8}$$

Question 16

(a) $P(7) = \dfrac{{}^{1}C_1 \times {}^{39}C_5}{{}^{40}C_6} = \dfrac{3}{20}$

$\therefore P(\bar{7}) = \dfrac{17}{20}$ ✓

i) $P(X=2) = {}^{5}C_2 \left(\frac{3}{20}\right)^2 \left(\frac{17}{20}\right)^3$

$= 0.1382 \quad (4 \text{ d.p.})$ ✓

ii) $P(X \geq 2) = 1 - P(X<2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[{}^{5}C_0 \left(\frac{3}{20}\right)^0 \left(\frac{17}{20}\right)^5 + {}^{5}C_1 \left(\frac{3}{20}\right)^1 \left(\frac{17}{20}\right)^4\right]$

$= 0.1648 \quad (4 \text{ d.p.})$ ✓

iii)

$P(7 \mid \text{five of the numbers 1 to 6}) = \dfrac{{}^{1}C_1 \times {}^{6}C_5}{{}^{40}C_6} = \dfrac{1}{639730}$ ✓

$P(E) = 1 - P\left(\begin{array}{c}\text{7 being the highest number in} \\ \text{none of the 5 games}\end{array}\right)$

$= 1 - {}^{5}C_0 \left(\frac{1}{639730}\right)^0 \left(\frac{639729}{639730}\right)^5$

$= 7.82 \times 10^{-6}$ (3 sig fig) ✓

Question 16

(b) (i) If $\angle CDB = \alpha$

then $\angle DAC = \alpha$ (Angle in the alternate segment theorem)

and $\therefore \angle DBC = \alpha$ (Angles on the circumference standing on the same arc are equal)

$\therefore \triangle BCD$ is isosceles (Base $\angle$'s equal).

(ii) $CD^2 = AC \cdot CK$ (Square on tangent).

but $CD = BC$ (Equal sides of isosceles $\triangle BCD$)

$\therefore BC^2 = AC \cdot CK$

$\therefore BC$ must be a tangent to circle AKB.

(c) (i) Consider first

$1 - t^2 + t^4 - t^6 + \dots + t^{4n}$ which is a G.P with $a=1$, $r=-t^2$ and $2n+1$ terms ✓

$S = \dfrac{a(1-r^n)}{1-r}$

$= \dfrac{1[1-(-t^2)^{2n+1}]}{1+t^2}$ ✓ $\quad (-t^2)^{2n+1} = (-1)^{2n+1} (t^2)^{2n+1}$

$= \dfrac{1+t^{4n+2}}{1+t^2} > \dfrac{1}{1+t^2}$ as $1+t^{4n+2} > 1$ for $t>0$.

$\therefore \dfrac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + \dots + t^{4n}$

Consider $1 - t^2 + t^4 - t^6 + \dots + t^{4n} - t^{4n+2}$ $\quad a=1$, $r=-t^2$ with $2n+2$ terms

$= \dfrac{a(1-r^n)}{1-r}$ ✓

$= \dfrac{1[1-(-t^2)^{2n+2}]}{1+t^2}$

$= \dfrac{1-t^{4n+4}}{1+t^2} < \dfrac{1}{1+t^2}$ as $1-t^{4n+4} < 1$ for $t>0$

So $1 - t^2 + t^4 - t^6 + \dots + t^{4n} - t^{4n+2} < \frac{1}{1+t^2}$

$\therefore \quad S < \frac{1}{1+t^2} + t^{4n+2}$

$\therefore \frac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + \dots + t^{4n} < \frac{1}{1+t^2} + t^{4n+2}$

ii) $\int_0^x \frac{1}{1+t^2}\,dt < \int_0^x 1 - t^2 + t^4 - t^6 + \dots + t^{4n}\,dt < \int_0^x \frac{1}{1+t^2} + t^{4n+2}\,dt$

$\left[\tan^{-1} t\right]_0^x < \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^6}{6} + \dots + \frac{t^{4n+1}}{4n+1}\right]_0^x < \left[\tan^{-1} t + \frac{t^{4n+3}}{4n+3}\right]_0^x$

$\therefore \tan^{-1} x - \tan^{-1} 0 < \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^6}{6} + \dots + \frac{x^{4n+1}}{4n+1}\right) - (0)$ ✓ progress

$< \left(\tan^{-1} x + \frac{x^{4n+3}}{4n+3}\right) - 0$

$\therefore \tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^6}{6} + \dots + \frac{x^{4n+1}}{4n+1} < \tan^{-1} x + \frac{x^{4n+3}}{4n+3}$

iii) as $n \to \infty$, $\frac{x^{4n+3}}{4n+3} \to 0$ as $0 \le x \le 1$.

$\therefore \quad x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ has an upper and lower bound of $\tan^{-1} x$. ✓

$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

iv) let $x = 1$ $\therefore \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots)$ ✓

$\therefore \pi = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$